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EXACT SOLUTIONS OF THE LINEAR PROBLEM OF THE STEADY-STATE WAVES CREATED BY A DIPOLE IN A FLOW OF A STRATIFIED FLUID*

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The problem of the steady-state waves which are formed when there is uniform flow of a non-viscous, incompressible, vertically stratified fluid round a dipole is considered in a linear formulation. Using the analytical properties of the solutions, two formulae are obtained for the vertical displacement field in the form of series of single integrals taken over the spectral curves. These formulae are simpler than those which have been previously proposed /1/ since the integrands do not contain special functions with logarithmic singularities and enable one to simplify the numerical analysis of the close domain of the wave field in which the asymptotic forms /2-4/ are applicable /5/.

1. Let a uniform flow of a non-viscous incompressible fluid of finite depth h occur around a finite dipole which is oriented in an antiparallel direction to the current. The density of the unperturbed fluid $\rho_0(z)$ depends on a single vertical coordinate z and does not decrease with depth. In the linear formulation, the field of the vertical displacements of the fluid particles $\zeta(x, y, z)$, which is formed by the dipole, is described by an equation with the boundary conditions

$$D^2\frac{\partial}{\partial z}\left(\rho_0\frac{\partial\zeta}{\partial z}\right)+\rho_0(N^2+D^2)\Delta_2\zeta=Mc^{-1}D^2\left\{\delta(x)\delta(y)\frac{\partial}{\partial z}[\rho_0\delta(z-z_0)]\right\} \tag{1.1}$$

$$\left(D^2\frac{\partial}{\partial z}-g\Delta_2\right)\zeta=0\quad(z=0),\quad \zeta=0\quad(z=-h)$$

$$\left(D=c\frac{\partial}{\partial x},\quad \Delta_2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},\quad N^2(z)=-g\rho_0^{-1}\frac{\partial\rho_0}{\partial z}\right)$$

Here, x and y are the horizontal coordinates, the fluid flows with a velocity c in the positive direction of the horizontal x-axis, the dipole is located at a point with the coordinates $(0, 0, z_0)$, M is the moment of the dipole, $N(z)$ is the Vaisala-Brunt frequency, $\delta(\cdot)$ is a delta function and g is the acceleration due to gravity. The radiation condition i.e. that the basic wave perturbations are formed lower down along the flow, has to be added to the boundary conditions (1.2).

By using forward and inverse Fourier transformations with respect to the horizontal coordinates, the solution of problem (1.1) can be written in the form of a double integral

$$\zeta(x, y, z)=Mc^{-1}(2\pi)^{-2}\rho_0(z_0)\operatorname{Re}\int_{-\pi/2}^{\pi/2}\varphi_-\,d\theta \tag{1.2}$$

$$\varphi_-=\int_{\sigma_-}\exp[i\beta^{1/2}\mu]\bar{\zeta}\,d\beta,\quad \mu=x\cos\theta+y\sin\theta \tag{1.3}$$

$$\bar{\zeta}(z, z_0; \lambda, \beta)=-\frac{\partial}{\partial z_0}\bar{\zeta}_0(z, z_0; \lambda, \beta),\quad \lambda=(c\cos\theta)^{-2}$$

where $\bar{\zeta}_0$ is Green's function for the problem

$$L(\eta)\equiv\frac{d}{dz}\left(\rho_0\frac{d\eta}{dz}\right)+\rho_0(N^2\lambda-\beta)\eta=0 \tag{1.4}$$

$$\frac{d\eta}{dz}-g\lambda\eta=0\quad(z=0),\quad \eta=0\quad(z=-h)$$

The arithmetic branch of the root β is selected in (1.3) and the integration of σ_- is carried out along the real axis from zero to infinity passing around the poles of $\bar{\zeta}$ from below along small semicircles in the complex plane of the parameter β with a cut $(-\infty, 0]$. This method of passing around the poles is a consequence of the radiation condition.

It is known from the theory of linear differential operators /6/ that $\bar{\zeta}$ is a meromorphic function of the parameters λ and β which, for each real λ, has a denumerable set, which is bounded from above, of just simple and real poles $\beta=\beta_n$, $\beta_1>\beta_2>\ldots$ The properties of the

*Prikl. Matem. Mekhan., 54, 6, 972-977, 1990 dispersion dependences $\beta_n(\lambda)$ have been described in /1, 4/. For each real β, the function $\bar{\zeta}$ has not more than a denumerable set, which is bounded from below, of simple and real poles with respect to λ. The singularities of $\bar{\zeta}$ are the eigenvalues of problem (1.4). Henceforth, we shall assume that β is a spectral parameter and we shall denote the eigenfunctions, corresponding to the values $\beta = \beta_n$, as $\eta = \eta_n(z;\lambda)$ where

$$\int_{-h}^{0} \rho_0 \eta_n^2\, dz = 1, \quad \int_{-h}^{0} \rho_0 \eta_n \eta_m\, dz = 0, \quad n \neq m$$

The function $\bar{\zeta}(z, z_0; \lambda, \beta)$ which is meromorphic with respect to β can be expanded in the simplest fractions /7/. The integrals of the individual terms of the corresponding expansion of (1.3) are expressed in terms of well-known special functions. As a result the accurate solution of problem (1.1) is written in the form of a sum of single integrals /1/

$$\zeta = M(2\pi c)^{-1} \rho_0(z_0) \sum_{n=1}^{\infty} \zeta_n, \quad \zeta_n = \zeta_{n1} + \zeta_{n2} \tag{1.5}$$

$$\zeta_{n1} = -\operatorname{Im} \int_{\theta_0}^{\pi/2} \exp(i\beta_n^{1/2}\mu)\, \psi_n\, d\theta, \quad \zeta_{n2} = \frac{1}{\pi} \operatorname{Re} \int_{-\pi/2}^{\pi/2} G(\beta_n^{1/2} |\mu|)\, \psi_n\, d\theta$$

$$\psi_n(z, z_0; \theta) = \eta_n(z, \lambda) \frac{d}{dz} \eta_n(z_0, \lambda)$$

$$G(u) = [\pi/2 - \operatorname{Si}(u)] \sin u - \operatorname{Ci}(u) \cos u, \quad |\arg u| < \pi$$

$\operatorname{Si}(u)$ and $\operatorname{Ci}(u)$ are sine and cosine integrals and θ_0 is the zero of the expression $\mu(\theta)$ in the interval $|\theta| \leqslant \pi/2$.

2. In deriving new expressions for $\zeta_n(x, y, z)$, we shall start out from a representation of the wave field in the form of a double integral (1.2), (1.3). We will first derive the formula which relates $\zeta(-x, y, z)$ and $\zeta(x, y, z)$. If the change of variables $\theta = -\theta_1$ is made in the integral (1.2) and the complex conjugate of (1.3) is taken, we obtain that

$$\zeta(-x, y, z) = Mc^{-1}\rho_0(z_0)(2\pi)^{-2} \operatorname{Re} \int_{-\pi/2}^{\pi/2} \varphi_+\, d\theta \tag{2.1}$$

and the functions φ_+ is given by formula (1.3) with the integration path, σ_-, replaced by the complex conjugate. Let this path be σ_+ which passes around the poles of $\bar{\zeta}$ along small semicircles from above in the complex plane of β. We now note that

$$\varphi_- = \varphi_+ - 2\pi i \sum_{n,\, \beta_n > 0} \exp(i\beta_n^{1/2}\mu)\, \psi_n \tag{2.2}$$

The summation in (2.2) is carried out over all the poles of $\bar{\zeta}$ which are positive along β. As a result, we find that the expression for $\zeta(x, y, z)$ can be written in the form

$$\zeta(x, y, z) = \zeta(-x, y, z) - M(2\pi c)^{-1} \rho_0(z_0) \operatorname{Im} \sum_{n=1}^{\infty} \int_{-\pi/2}^{\pi/2} \exp(i\beta_n^{1/2}\mu)\, \psi_n\, d\theta \tag{2.3}$$

This formula enables one to confine oneself to the consideration of the case when $x > 0$.

Next, let λ be a complex number, β_0 be any pole of $\bar{\zeta}$ and η_0 be the corresponding eigenfunction of problem (1.4). By multiplying the equality $L(\eta_0) = 0$ by the function η_0^*, which is the complex conjugate of η_0, and integrating the resulting expression along z from $-h$ to 0 using the boundary conditions from (1.4), we obtain the integral identity

$$\lambda \left[g\rho_0 |\eta_0|_{z=0}^2 + \int_{-h}^{0} \rho_0 N^2 |\eta_0|^2\, dz \right] = \int_{-h}^{0} \rho_0 \left[\left| \frac{d\eta_0}{dz} \right|^2 + \beta_0 |\eta_0|^2 \right] dz$$

from which it follows that, when $\operatorname{Im} \lambda \neq 0$,

$$\operatorname{Im} \lambda \cdot \operatorname{Im} \beta_0 > 0 \tag{2.4}$$

and, when $\operatorname{Im} \lambda = 0$ and $\operatorname{Re} \lambda \leqslant 0$, the function $\bar{\zeta}$ can only have negative real poles along β.

Inequality (2.4) also remains true in the case when β_0 is a fixed complex number, $\operatorname{Im} \beta_0 \neq 0$ and λ is a pole of $\bar{\zeta}$.

Let us now transpose expression (2.1). In order to do this, we consider the function

$\varphi(\theta)$ specified by formula (1.3) in which the integration path goes from 0 to ∞ along the real axis of β. By taking account of the inequality (2.4), we find that $\varphi(\theta)$ is analytical in the domains where $\operatorname{Im}\lambda \neq 0$ and $\operatorname{Im}\lambda = 0$, $\operatorname{Re}\lambda \leqslant 0$ and $\operatorname{Re}[i\mu(\theta)] < 0$. The half band in the complex plane of θ: $-\pi/2 \leqslant \operatorname{Re}\theta < 0$, $\operatorname{Im}\theta > 0$ satisfies these conditions. The function $\bar{\zeta}$ is bounded when $\operatorname{Im}\theta \to +\infty$ and the values of β are real, which follows from the asymptotic behaviour of Green's function /6/. The above-mentioned properties of $\bar{\zeta}$ and φ enable one, using contour integration, to carry out the transformation

$$J_1 = \int_{-\pi/2}^{0} \varphi_+ \, d\theta = \int_{-\pi/2}^{-\pi/2+i\infty} \varphi \, d\theta + \int_{-0+i\infty}^{-0} \varphi \, d\theta \tag{2.5}$$

The function $\varphi(\theta)$ is also analytic in the semiband $\gamma < \operatorname{Re}\theta \leqslant \pi/2$, $\operatorname{Im}\theta < 0$, $\gamma = \arcsin(y/\sqrt{x^2+y^2})$. In the interval l: $\operatorname{Re}\theta = \pi/2$, $\operatorname{Im}\theta \leqslant \alpha < 0$, the function $\bar{\zeta}$ only has real negative poles along β and, therefore, $\varphi_+(\theta) = \varphi(\theta)$ in l. Furthermore, $0 < \arg[\mu(\theta)] < \pi/2$ in this interval which enables one to rotate the contour of integration in the expression for the function $\varphi(\theta)$ into the upper complex halfplane of the parameter β. As a result, we find that

$$\varphi(\theta) = \int_{+i0}^{-\infty+i0} \exp(i\beta^{1/2}\mu)\bar{\zeta}\, d\theta \tag{2.6}$$

Here, the integration is carried out along the upper bank of the cut $(-\infty, 0]$. In representation (2.6), the function $\varphi(\theta)$ allows analytical continuation into the domain where $\operatorname{Re}[\mu(\theta)] > 0$ and $\operatorname{Im}\lambda \leqslant 0$. We now carry out the transformation

$$J_2 = \int_{0}^{\pi/2} \varphi_+ \, d\theta = \int_{\pi/2-i\infty}^{\pi/2} \varphi \, d\theta + \int_{0}^{-i\infty} \varphi \, d\theta \tag{2.7}$$

using contour integration.

The integrals from the right-hand sides of (2.5) and (2.7) are now added in a pairwise manner. It can be shown that the real part of the sum of the first of them is equal to zero and they therefore make no contribution to (2.1). So, we have

$$\operatorname{Re}(J_1 + J_2) = \operatorname{Re}\Big[\int_{-0+i\infty}^{-0} \varphi \, d\theta + \int_{0}^{-i\infty} \varphi \, d\theta\Big] \tag{2.8}$$

The integrand of the first integral in (2.8) can be represented in the form of (2.6). Actually, here, for values of θ located to the left of the positive part of the imaginary axis, $\operatorname{Im}\lambda < 0$, by virtue of (2.4), the function $\bar{\zeta}$ has no poles along β when $\operatorname{Im}\beta \geqslant 0$ and in the neighbourhood of the integration path of the interval $0 < \arg[\mu(\theta)] < \pi/2$ which is being considered. Next, since $\operatorname{Im}\beta > 0$ in (2.6), the function $\bar{\zeta}$ has no poles along λ in the domain $\operatorname{Im}\lambda \leqslant 0$ and integration with respect to θ in (2.8) can therefore be carried out directly along the imaginary axis. By adding the first integral from (2.8), which has been transformed in this manner, to the complex conjugate of the second integral and taking account of the value of $\arg\beta^{1/2}$ on the banks of the cut, we find that

$$\operatorname{Re}(J_1 + J_2) = 2\pi \operatorname{Im} \int_{0}^{i\infty} \sum_{n,\, \beta_n < 0} \exp(-|\beta_n|^{1/2}\mu)\, \psi_n \, d\theta$$

Here, the contributions from the poles of $\bar{\zeta}$ which are negative with respect to β are summed.

As a result, the double integrals from (2.1) are transformed into the sum of single integrals

$$\zeta = M(2\pi c)^{-1}\rho_0(z_0)\sum_{n=1}^{\infty}\zeta_n, \quad \zeta_n = \zeta_{n3} + \zeta_{n4} \tag{2.9}$$

$$\zeta_{n3} = -H(x)\operatorname{Im}\int_{-\pi/2}^{\pi/2}\exp(i\beta_n^{1/2}\mu)\,\psi_n\, d\theta$$

$$\zeta_{n4} = \operatorname{Im}\int_{0}^{i\infty} H(-\beta_n)\exp(-|\beta_n|^{1/2}\mu)\,\psi_n\, d\theta$$

where $H(\cdot)$ is the Heaviside unit function.

The parameter $\lambda = (c\cos\theta)^{-2}$ only takes positive values along the integration paths in (2.9). It is known that, for real λ, the dispersion dependence $\beta = \beta_n(\lambda)$, which is a monotonically increasing function of λ, only has a single zero at the point $\lambda = \lambda_n$, $\lambda_n = c_n^{-2}$, where c_n is the velocity of propagation of the long waves of the n-th mode. In the subcritical case $c < c_n$, the function β_n is positive in the interval $-\pi/2 < \theta < \pi/2$ and, in the interval $(0, i\infty)$, there is just a single zero at the point $\theta = \theta_n$, $\theta_n = i\ln(\mathrm{fr}_n^{-1} + \sqrt{\mathrm{fr}_n^{-2} - 1})$ where $\mathrm{fr}_n = c/c_n$ is the Froude number of the n-th mode. Hence, when $c < c_n$, the integration of ζ_{n4} actually begins from the point $\theta = \theta_n$. In the supercritical case when $c > c_n$, the function β_n has two zeros at $\theta = \pm\theta_n$, $\theta_n = \arccos(\mathrm{fr}_n^{-1})$ in the interval $(-\pi/2, \pi/2)$ and does not have any zeros on the imaginary axis θ. Since only the imaginary part of the integral is required in the formula for ζ_{n3}, the interval $(-\theta_n, \theta_n)$ makes no contribution to ξ_{n3}.

We note that, for computations using formulae (2.9), calculations of the dispersion dependences and the eigenfunctions of the Strum-Liouville problem corresponding to (1.4) are only required for real non-negative values of λ.

The parameter λ also takes real, non-negative values on the straight lines $\mathrm{Re}\,\theta = \pm\pi/2$ in the complex plane of the parameter θ. In a similar way to that employed in deriving (2.9), it is possible to derive a further representation of the solution

$$\zeta = M(2\pi c)^{-1}\rho_0(z_0)\sum_{n=1}^{\infty}\zeta_n, \quad \zeta_n = \zeta_{n5} + \zeta_{n6} \tag{2.10}$$

$$\zeta_{n5} = -\mathrm{Im}\int_{-\pi/2}^{\theta_{n1}} \exp(i\beta_n^{1/2}\mu)\psi_n\,d\theta, \quad \zeta_{n6} = \mathrm{Im}\int_{\pi/2}^{\pi/2+i\infty} \exp(-|\beta_n^{1/2}|\mu)\psi_n\,d\theta$$

Here, $\theta_{n1} = -\theta_n$ in the case when $c > c_n$, and $\theta_{n1} = \theta_n$ when $c \leqslant c_n$. Integration with respect to θ when $c < c_n$ in the formula for ζ_{n5} is carried out in the intervals $(-\pi/2, 0)$ and $(0, \theta_n)$.

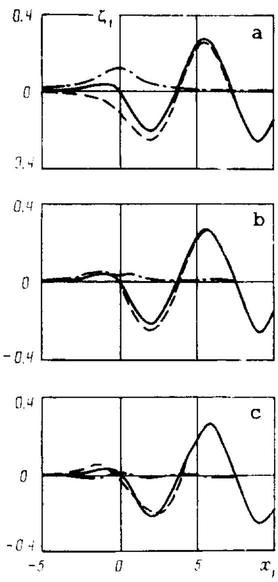

Fig.1

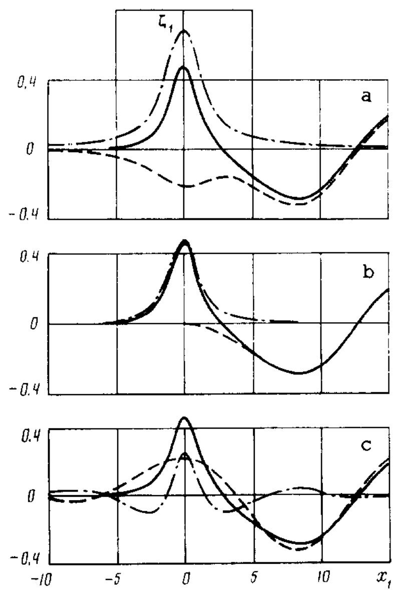

Fig.2

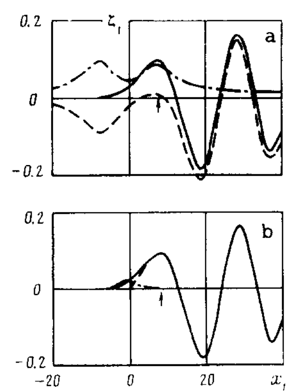

Fig.3

The solutions (2.9) and (2.10) which have been obtained in this paper are of a single type with the integrals occurring in them differing in their integration limits. The new formula are more convenient than (1.5) for numerical implementation since the integrands in (2.9) and (2.10) do not contain special functions and do not have logarithmic singularities. We further note that the limits of integration in (2.9) and (2.10) are independent of the spatial coordinates (x, y) unlike the integrals of ζ_{n1}.

Example. Let us now consider the case of uniform stratification ($N^2 = \mathrm{const}$). The Boussinesq approximation and the "solid cover" condition on the surface of the fluid enable us to obtain explicit expressions for the eigenvalues and eigenfunctions

$$\beta_n = (c_1^2\lambda - n^2)\left(\frac{\pi}{h^2}\right), \quad \psi_n = \frac{2\pi n}{h^2}\sin\frac{\pi n z}{h}\cos\frac{\pi n z_0}{h}, \quad c_1 = \frac{Nh}{\pi}$$

The results of calculations of the first mode ζ_1 and its terms using formulae (1.5) (a), (2.9) (b) and (2.10) (c) are shown in Fig.1 for the subcritical case when $\mathrm{fr}_1 = c/c_1 = 0.8$ and, in Fig.2, for the supercritical case when $\mathrm{fr}_1 = 2.0$. The values of the integrals are given to within a factor of ψ_n which, in this example, is independent of the variable θ. ζ_{11}, ζ_{13} and ζ_{15} and represented by broken lines, ζ_{12}, ζ_{14} and ζ_{16} are denoted by the dotted and dashed lines and ζ_1 is represented by the solid line. The calculations were carried out for $y = 1 \cdot h/\pi$ and, in the figures, $x_1 = x\pi/h$.

The integrals ζ_{12}, ζ_{14} and ζ_{16} are even functions of the x coordinates. At small distances along y from the wave generator, their contribution is considerable in the neighbourhood of the y-axis while, in the downstream domain (when $x > 0$), the first terms of ζ_1 make the main contribution to the wave field.

The dependence of the integrals being considered on x in the neighbourhood of the leading front /3/ when $\mathrm{fr}_1 > 1$ and at large values of y is of interest. The calculations which were carried out show that the nature of the dependence of ζ_{14} and ζ_{16} does not change as y increases. The maximum in ζ_{12} divides into two as y increases. One of these is localized in the neighbourhood of the leading front and the second is symmetrical to it with respect to the y-axis /1/. Results of the calculations for $y = 5h/\pi$ and $\mathrm{fr}_1 = 2.0$ using formulae (1.5) and (2.9) (b) are shown in Fig.3. The notation on the curves is the same as that in Figs.1 and 2. Calculations using the third formula (2.10) showed that, for these values of the parameters, the magnitudes of ζ_{16} are less than 5% of ζ_{15}. The position of the leading front is indicated by the arrows in the figure. We note that, in the region where $x < 0$, the term ζ_{11} compensates ζ_{12}. The same feature may also be noted at a small value of y in Fig.2,a.

An analysis of the calculations shows that, for a given accuracy of the calculations, the contribution of the terms ζ_{n4} and ζ_{n6} from formulae (2.9) and (2.10) can be taken into account in a smaller domain than the contribution to ζ_{n2} from (1.5). For instance, in the interval from $y = 2h$ to $y = 10$ to $20h$, starting from which the asymptotic forms /5/ can be used, the characteristics of the waves are estimated with an accuracy of 5% using the integrals ζ_{n3} and ζ_{n5}.

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